Spanning trees in complete uniform hypergraphs and a connection to r-extended Shi hyperplane arrangements

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Abstract

We give a Cayley type formula to count the number of spanning trees in the complete r-uniform hypergraph for all $r \geq 3$. Similar to the bijection between spanning trees of the complete graph on $(n+1)$ vertices and *Parking functions* of length n, we derive a bijection from spanning trees of the complete $(r+1)$-uniform hypergraph which arise from a fixed r-perfect matching (see Section 2) and r-*Parking functions* of an appropriate length. We observe a simple consequence of this bijection in terms of the number of regions of the r-extended Shi hyperplane arrangement in n dimensions, S_n^r.

1 Introduction

We give a formula to count the number of spanning trees in the complete r-uniform hypergraph for all $r \geq 3$ (we call them r-spanning trees). We first present the case when $r = 3$ where we use the Pfaffian Matrix Tree Theorem of Masbaum and Vaintrob [MV-02]. Using ideas from that proof, we present our result for $r \geq 4$.

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. 3-spanning trees are defined in the following way. Let $H = (V, \mathcal{F})$ be a 3-uniform hypergraph. Consider the bipartite graph with V on one side, and $\mathcal{F}$ on the other. Each hyperedge $e = \{a, b, c\}$ is connected to precisely the vertices a, b and c. If the bipartite graph on some r hyperedges $E = \{e_1, e_2, \ldots, e_r\}$ and all the vertices V is a tree in the graph theoretic sense, then we call the hypergraph $H = (V, E)$ as a 3-spanning tree. Figure 1 shows a 3-spanning tree $T = ([7], \{123, 347, 356\})$ on 7 vertices where 123 is an abbreviation for the 3-hyperedge $\{1, 2, 3\}$ and so on. We prove the following theorem about 3-spanning trees.

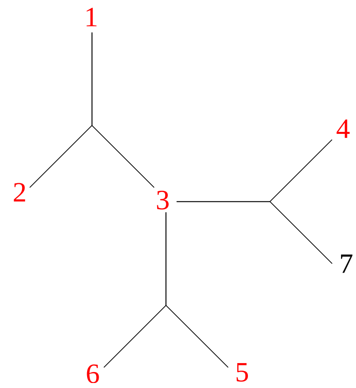

Figure 1: A 3-uniform spanning tree on 7 vertices

Theorem 1 *The number of 3-spanning trees of the complete 3-uniform hypergraph on* $n = 2k+1$ *vertices is* $1 * 3 \cdots * (2k-1) * (2k+1)^{k-1}$.

By counting the number of edges in the bipartite graph representation of a 3-spanning tree in two ways, it is clear that for spanning trees to exist in a 3-uniform hypergraph on n vertices, n has to be odd. The same double counting argument shows that the number of hyperedges in any 3-spanning tree on $n = 2k+1$ vertices is k. Thus Theorem 1 is similar to Cayley's Theorem for counting spanning trees of the complete graph on n vertices. If we interpret the term occuring in Cayley's Theorem as v^{e-1} with v being the number of vertices spanned and e being the number of edges in any spanning tree, we see that Theorem 1 has a similar term $(2k+1)^{k-1}$. There is an additional multiplicative term of $1 * 3 * \cdots * (2k-1)$. This term is the number of perfect matchings in the complete graph on $n-1 = 2k$ vertices and from the proof of Theorem 1, one sees why this term arises.

Let $r \geq 4$ be a positive interger. r-spanning trees in r-uniform hypergraphs are defined analogously. Figure 4 shows a 4-spanning tree. Henceforth, when we talk about r-spanning trees, we omit mentioning that the underlying graph is the complete r-uniform hypergraph. We prove a similar result about the number of r-spanning trees on $n = (r-1)k+1$ vertices. For $r \geq 3$, it can be checked that the number of $(r-1)$-perfect matchings of the complete $(r-1)$-uniform hypergraph on $(r-1)k$ vertices is $rPM = \binom{(r-1)\cdot k-1}{r-2} * \binom{(r-1)\cdot(k-1)-1}{r-2} * \cdots * \binom{r-2}{r-2}$. The proofs of both Theorems 1 and 2 are in Section 2.

Theorem 2 *For* $r \geq 3$*, the number of* r*-spanning trees on* $n = (r-1)k+1$ *vertices is* $rPM * n^{k-1}$.

In Section 3, we give an exponential generating function for the number of rooted r-spanning trees on $[n]$. These are analogs of the famous relation $D(x) = x\exp(D(x))$ where $D(x)$ is the exponential generating function for rooted spanning trees. Though one can derive Theorems 1 and 2 from this (ie the total count), we present a non generating function proof of both these theorems in Section 2 as we get extra information about the number of r-spanning trees on $[n]$, when we fix an $(r-1)$-perfect matching on $n-1$ vertices. This count of a subset of the set of all r-spanning trees will be used in Sections 4 and 5.

In Section 4, we give a bijection between r-parking functions of length k and $(r+1)$-spanning trees on $rk+1$ vertices, which arise from a fixed r-perfect matching on rk vertices. We recall the definition of *parking functions* of length k. There are k cars $1, 2, \ldots, k$ and k parking spaces $0, 1, \ldots, k-1$, in this order. All cars enter the street at the end close to the zeroth parking slot. They enter in increasing order and each car i has its preferred parking slot a_i. Car i drives to slot a_i and parks there if that slot is free. If not, it tries the next higher parking slot and so on till it gets an empty parking slot. A sequence $\overline{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$ is said to be a *parking function* of length k if all cars are able to park following the above rules. Spanning trees of the complete graph on $k+1$ vertices are in a bijection with *parking functions* of length k (see [EC2]). There is an alternative characterization of parking functions. Let $\overline{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{Z}_{\geq 0}^k$ and let $\overline{b} = (b_1, b_2, \ldots, b_k)$ where $b_1 \leq b_2 \leq \cdots \leq b_k$ be the weakly increasing rearrangement of $\overline{a}$. Then $\overline{a}$ is a parking function iff $b_i \leq (i-1)$ for all i. This algebraic definition of Parking functions was generalised in the following manner to yield r-parking functions for a positive integer r (see [St-98]). A sequence $\overline{a} \in \mathbb{Z}_{\geq 0}^k$ is called an r-parking function iff its weakly increasing rearrangement $\overline{b}$ satisfies $b_i \leq r(i-1)$ for all i. For $r, k \geq 1$, let Park_k^r be the set of r-parking functions of length k. We prove the following theorem.

Theorem 3 *For all $r \geq 1$, there is a bijection between the set of $(r+1)$-spanning trees on $n = rk+1$ vertices which arise from a fixed r-perfect matching and the set Park_k^r.*

Our bijection is very similar to that of Chebekin and Pylyavskyy [CP-05] and uses the *breadth-first search* order of vertices in a rooted r-spanning tree.

In Section 5, we point out a simple connection to the number of regions of the r-extended Shi hyperplane arrangement in m dimensions, denoted S_m^r. This connection is a consequence of Theorem 3 and Theorem 2.1 in [St-98]. We recall the definition of S_m^r. It is given by the following set of hyperplanes in $\mathbb{R}^m$

$$x_i - x_j = -r+1, -r+2, \ldots, r, \text{ for } 1 \leq i < j \leq m$$

It has $\binom{m}{2} * 2r$ hyperplanes. The number of regions of the 1-extended Shi hyperplane arrangement in n dimensions is identical to the number of spanning trees of the complete graph on $n+1$ vertices (see [St-98]). We prove an analog of this for higher values of r.

Theorem 4 *Let $r, k \geq 1$. The number of regions of the r-extended Shi arrangement S_k^r is equal to the number of $(r+1)$-spanning trees on $n = rk+1$ vertices arising from a fixed r-perfect matching on $n-1$ vertices.*

2 Counting r-spanning trees

We first prove our result for 3-spanning trees and then for r-spanning trees where $r \geq 4$.

2.1 Counting 3-spanning trees

We review briefly the theorem of Masbaum and Vaintrob [MV-02] where they enumerate with a $\pm$ sign all 3-spanning trees of a 3-uniform hypergraph. We only need the case when the 3-uniform hypergraph is complete. As remarked, for 3-spanning trees to exist in 3-uniform hypergraphs, the number of vertices has to be odd. Let $n = 2k+1$ and let ST_n be the set of 3-spanning trees on n vertices. For $i, j, k \in [n]$, all three indices being distinct, let $x_{i,j,k}$ be a variable with the following 'sign' property. For all $\sigma \in S_3$, $x_{\sigma(i),\sigma(j),\sigma(k)} = \text{sign}(\sigma)x_{i,j,k}$.

Associate an $n \times n$ matrix $A = (a_{i,j})$ with the complete 3-uniform hypergraph where $a_{i,j} = \sum_{k=1;k\neq i,j}^{n} x_{i,j,k}$. Due to the sign property, this matrix is skew-symmetric. Let A_i be the submatrix of A obtained by omitting the i-th row and the i-th column.

The theorem of Masbaum and Vaintrob says that the pfaffian of A_i gives a ± 1 signed enumeration of all 3-spanning trees $T \in ST_n$. The coefficient ± 1 arises dues to two multiplicative factors: a ± 1 sign denoted $\mathsf{sign}(T)$ which is the sign of a permutation π of the vertices obtained by embedding T in the plane; and a product of $x_{i,j,k}$'s for each hyperedge $\{i, j, k\}$ of T. Each hyperedge variable $x_{i,j,k}$ has to be ordered acording to π. Due to the 'sign property', each such hyperedge might get a ± 1. Hence multiplication yields an overall ± 1 coefficient. One thus gets a term y_T with a ± 1 coefficient for each 3-spanning tree T. See Reiner and Hirschman [RH-02] for an exact procedure to obtain y_T from T.

Let PM_{2k} be the set of perfect matchings in the complete graph on $2k$ vertices. Clearly, $|PM_{2k}| = 1 * 3 * \cdots * (2k-1)$. It is known that the pfaffian of a skew symmetric $2k \times 2k$ matrix A of is given by the expression (see [RH-02])

$$Pf(A) = \sum_{M \in PM_{2k}} (-1)^{\text{cross}(M)} \prod_{i,j \in M, i<j} a_{i,j} \tag{1}$$

where cross(M) is the number of edges which cross the perfect matching M. Formally, $\text{cross}(M) = |\{i < j < k < l : \{i, k\}, \{j, l\} \in M\}|$.

In the above expression, we call the terms arising from a fixed perfect matching M as the "terms of M in the pfaffian expansion". The Pfaffian Matrix Tree Theorem is:

Theorem 5 (Masbaum and Vaintrob [MV-02]) *Let A be the matrix defined above and let n be odd. For all $1 \leq i \leq n$,*

$$Pf(A_i) = (-1)^{i-1} \sum_{T \in ST_n} y_T$$

Proof: (Of Theorem 1)

Consider the matrix A_n obtained by deleting the n-th row and the n-th column of A and expand the pfaffian of A_n as in equation 1. We have terms for each perfect matching $M \in PM$. It is conceivable that some 3-spanning trees T may occur as terms of many perfect matchings, and they sum up nicely to yield a ± 1 coefficient for each such T. We show below that this cannot happen. This observation is what is used to generalise to get the result for the complete r-uniform hypergraph for $r \geq 4$.

We show that each 3-spanning tree appears exactly once in the pfaffian expansion of equation 1 in the term corresponding to some perfect matching. We recall that ST_n is the set of spanning trees of the complete 3-uniform hypergraph on n vertices, and PM_{2k} is the set of perfect matchings of the complete graph on $2k$ vertices.

Lemma 1 *There is a one-to-one mapping $f : ST_n \mapsto PM_{2k}$ such that for all $T \in ST_n$, the term y_T appears in the pfaffian expansion of A_n only among the terms of the perfect matching $f(T)$.*

Proof: We give an algorithm to obtain the mapping f. Let $T \in ST_n$ with $T = (V, E)$. Since T spans all the vertices, vertex n has to appear in at least one hyperedge. As each hyperedge has exactly 3 vertices, deleting the vertex n from all hyperedges of E will result in some hyperedges having size 2 and others retaining their size. (At least one hyperedge will have its size reduced by this procedure.) All the currently size 2 hyperedges are put in $f(T)$. The vertices newly added to $f(T)$ are termed "matched". If $f(T)$ is a perfect matching, then we are done. Otherwise, there still remain some hyperedges in E which have size 3. Delete all the *matched* vertices. This will again reduce the size of some hyperedges to 2. These size 2 hyperedges are added to $f(T)$ and we iterate. It is clear that since we obtain at least one matching edge in each iteration, that in at most k iterations, we will terminate with a perfect matching on the vertex set $[2k]$. It is also clear that we end up with exactly one perfect matching of $[2k]$. The perfect matching we construct has the property that from every hyperedge of T, we have 2 vertices matched. For example, on the spanning tree of Figure 1, the algorithm will output the perfect matching $\{\{3, 4\}, \{1, 2\}, \{5, 6\}\}$.

We need to check that y_T occurs in the pfaffian expansion corresponding to the perfect matching $f(T)$. This is easy as (ignoring the $\pm$ sign of y_T,) all hyperedges of T have exactly two vertices in the perfect matching $f(T)$. We also need to check that y_T does not occur among terms of any other perfect matching in the pfaffian expansion of A_n. To see this, we note that two vertices from all hyperedges in E need to occur in any perfect matching M' for y_T to occur among the terms of M' and that this happens only for $f(T)$.

∎

Remark 1 *It is easy to see from the above algorithm that each spanning tree T has at least one leaf (ie a hyperedge with two vertices of degree 1) which is matched in the perfect matching. This will be used in the bijection of Lemma 2. Suppose we call the edge of the perfect matching $e = \{a, b\}$, then we note that there is a unique hyperedge $f \in E(T)$ such that $f = \{a, b, \ell\}$ for some $\ell \in ([n] - \{a, b\})$.*

Remark 2 *From the above algorithm, it is clear that each edge $e = \{a, b\}$ of the perfect matching $f(T)$ is in a unique hyperedge $\{a, b, x\}$ of T.*

Since each 3-spanning tree T comes up in the pfaffian expansion in exactly one perfect matching, we infer that each 3-spanning tree occurs exactly once among the terms in the pfaffian expansion of A_n. Hence, only terms corresponding to non spanning trees (ie terms with cycles in it) get cancelled and Reiner and Hirschman [RH-02] exhibited a sign reversing

involution cancelling exactly these cyclic terms. Lemma 1 shows that each 3-spanning tree occurs in the pfaffian expansion in exactly one perfect matching. Below we count the number of 3-spanning trees t_M which get mapped to a fixed perfect matching M under f. Theorem 1 follows by adding t_M over all perfect matchings M in PM_{2k}.

Lemma 2 *Let $n = 2k+1$ and let $M \in PM_{2k}$. Under f, M gives rise to $(2k+1)^{k-1}$ 3-spanning trees on the vertex set $[n]$.*

Proof: We give a *Prufer* type bijection. We prove the lemma for the perfect matching $M = \{\{1,2\},\{3,4\},\ldots,\{2k-1,2k\}\}$. It will be clear that a similar proof works for other perfect matchings. We give a bijection between 3-spanning trees which arise from the perfect matching M to the set of all strings $\overline{s} = (s_1, s_2, \ldots, s_{k-1})$ with $k-1$ coordinates where for all $1 \le i \le k-1$, $1 \le s_i \le n$.

For one direction, given a 3-spanning tree T, by Remark 1, we know that there is at least one *leaf hyperedge* (ie a hyperedge which contains an edge e of M as degree one vertices; we refer to the edge $e \in M$ as a *leaf edge*). If there are many such leaf hyperedges, we choose a total order π on the *leaf edges* and pick the leaf hyperedge with the smallest (wrt π) leaf edge. For the remaining part of the proof, we use the total order $\{1,2\} <_\pi \{3,4\} <_\pi \cdots <_\pi \{2k-1,2k\}$ on the edges of M. We pick one leaf hyperedge in every iteration. Put $i=1$, let ℓ_i be the picked leaf hyperedge and let its lead edge be $e_i = \{a_i, b_i\}$. Let s_i be connection point of ℓ_i (ie, let $\ell_i = \{s_i, a_i, b_i\}$ is the unique hyperedge containing both a_i and b_i). Delete both the vertices a_i and b_i, increase i by 1 and repeat. Note that there will again exist at least one leaf hyperedge in the deleted subhypergraph. This gives us a sequence of $k-1$ numbers, each in the range 1 to n. For example, when we start with the tree in Figure 2, we get the sequence $\overline{s} = (3,3,4)$.

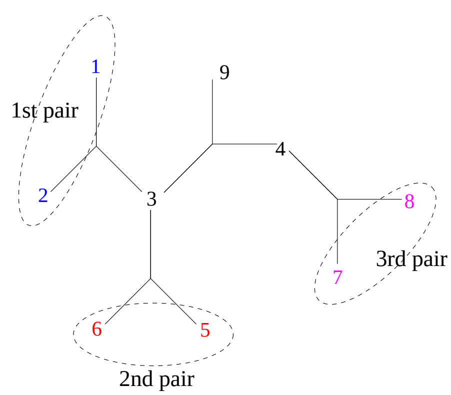

Figure 2: One direction of the Prufer type correspondence.

For the converse, we group the vertices of the graph into components based on the perfect matching (see Figure 3). Thus, we begin with $k+1$ components $\{1,2\},\{3,4\},\ldots,\{2k-1,2k\},n$. As shown in Figure 3, we need to assign values to the variables $x_1, x_2, \ldots, x_k$ such that the resulting 3-uniform hypergraph is a 3-spanning tree. If $1 \le a < n$ is a number, then

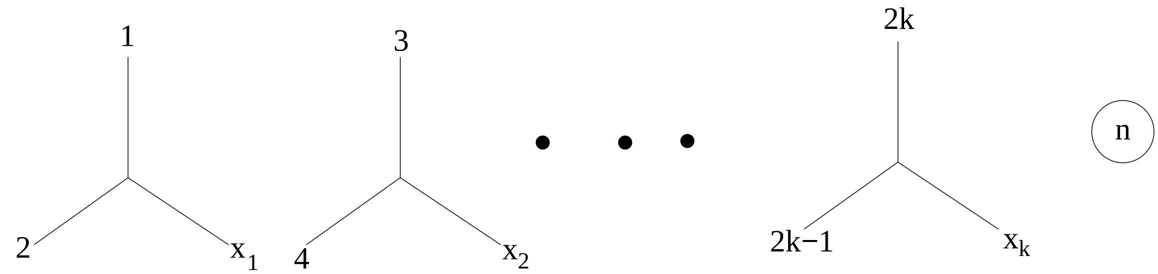

Figure 3: The components of M at the beginning.

let $m(a)$ be the edge of the perfect matching M which contains a (ie $m(5) = m(6) = \{5, 6\}$ and so on). If $a = n$, then $m(a)$ is undefined.

Let $\overline{s} = (s_1, s_2, \dots, s_{k-1})$ be the given sequence of numbers. We add hyperedges sequentially. Initially, all the components are marked "unfinished". Starting from $i = 1$, let b_i be a smallest edge of the perfect matching M which does not occur as $m(s_j)$ for $j \geq i$. We mark b_i as "finished", add the hyperedge $\{b_i, s_i\}$, increase i by 1 and iterate. It is easy to check that we end up with two unfinished components. We add them as a hyperedge. For example, the above procedure for $n = 9$, on the sequence $\overline{s} = (3, 3, 4)$ will yield the spanning tree of Figure 2. ■

The above proof clearly works for any perfect matching (we also need a total order between the edges of the perfect matching). Summing over all perfect matchings in PM_{2k} completes the proof of Theorem 1. ■

Remark 3 *A similar theorem is true for any 3-uniform hypergraph H, though the number of 3-spanning trees arising from a perfect matching M may depend on M and H.*

2.2 Counting r-spanning trees

In this subsection, we consider r-spanning trees where $r \geq 4$. It can be checked that r-spanning trees on n vertices exist only when $n \equiv 1 (mod\ r-1)$.

Let T be an r-spanning tree on $n = (r-1)k+1$ vertices. As before, the vertex n lies in at least one hyperedge. Using the same deletion process of Lemma 1, we see that T arises from exactly one $(r-1)$-perfect matching on the vertex set $[(r-1)k]$. We illustrate this on the 4-spanning tree T shown in Figure 4. The 3d matching we get from T is $\{3, 8, 9\}, \{1, 2, 5\}$ and $\{4, 6, 7\}$. Clearly each r-spanning tree T gives rise to one fixed $(r-1)$-perfect matching and we need to count the number of r-spanning trees which arise from a fixed $(r-1)$-perfect matching.

Proof: (Of Theorem 2) A *Prufer* type bijection works in this case as well to show that for a fixed $(r-1)$-perfect matching M on $[(r-1)k]$, the number of r-spanning trees that arise from it equals $(rk+1)^{k-1}$. ■

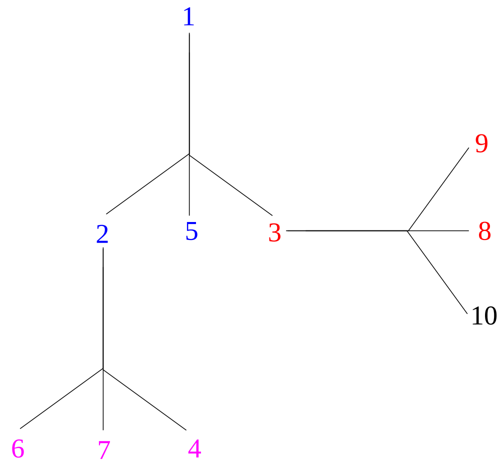

Figure 4: A spanning tree of the 4-uniform complete graph on 10 vertices.

3 Exponential generating functions

In this section, we present an exponential generating function for the number of rooted r-spanning trees. We begin with the case $r = 3$. Let T_n be the set of rooted 3-spanning trees on $[n]$ and let t_n be the number of rooted 3-spanning trees. We set $t_0 = 0; t_1 = 0$. We note that $t_n = 0$ for all even n. Let

$$T(x) = \sum_{n \geq 0} t_n \frac{x^n}{n!}$$

be the exponential generating function for the sequence t_n.

Let pm_k be the number of perfect matchings of the complete graph on $[k]$. Let

$$E_{pm}(x) = \sum_{n \geq 0} pm_n \frac{x^n}{n!}$$

be the exponential generating function of the sequence pm_k. We prove the following theorem.

Theorem 6 *The exponential generating functions $E_{pm}(x)$ and $T(x)$ satisfy*

$$T(x) = xE_{pm}(T(x)) \tag{2}$$

Proof: We give a 5-step procedure to build rooted 3-spanning trees on $[n]$. It is easy to see that Theorem 6 is equivalent to this procedure. To build a rooted 3-spanning tree T on $[n]$, we proceed as follows:

1. Pick rt in T to serve as its root.
2. Partition $[n] - \{rt\}$ into an even number of non empty blocks $B_1, B_2, \ldots, B_{2m}$.
3. Choose a rooted 3-spanning tree T_i on each of the blocks B_i for $1 \leq i \leq 2m$.

4. Choose a perfect matching $\{a_i, b_i\}$ for $i = 1, 2, \ldots, m$ of the roots of $T_1, T_2, \ldots, T_{2m}$.

5. Add m 3-hyperedges $\{rt, a_i, b_i\}$ for $i = 1, \ldots, m$.

■

One can infer Theorem 1 from the above by applying Lagrange Inversion Formula (see [EC2]). It is easy to see from Equation (2) that $t_n = 0$ for even values of n. Below, we compute t_n for odd n.

$$[x^{2k+1}]T(x) = \frac{1}{2k+1}[y^{2k}](E_{pm}(y))^{2k+1}$$

One can check that

$$[y^{2k}](E_{pm}(y))^{2k+1} = \left(\frac{2k+1}{2}\right)^k \frac{1}{k!}$$

Thus, $\frac{t_{2k+1}}{(2k+1)!} = \frac{1}{(2k+1)k!}\left(\frac{2k+1}{2}\right)^k$, or $t_{2k+1} = \frac{(2k)!}{k!2^k}(2k+1)^k = 1 \cdot 3 \cdots (2k-1)(2k+1)^k$. But t_{2k+1} is the number of rooted 3-spanning trees on $[2k+1]$ which is $(2k+1)$ times number of 3-spanning trees. That completes another proof of Theorem 1.

3.1 Exponential generating function for rooted r-spanning trees

For counting r-spanning trees, one modifies the Steps 2 onwards of the above procedure. To construct an r-spanning tree T on $n = (r-1)k + 1$ vertices, we use the procedure below.

1. Pick rt in T to serve as its root.

2. Partition $[n] - \{rt\}$ into m non empty blocks $B_1, B_2, \ldots, B_m$, where $m = p(r-1)$ for a positive integer p.

3. Choose a rooted r-spanning tree T_i on each of the blocks B_i for $1 \leq i \leq m$.

4. Group the $p(r-1)$ roots of $T_1, T_2, \ldots, T_m$ into p blocks each of size $r-1$. ie choose an $(r-1)$-perfect matching $R = \{r_1, r_2, \ldots, r_p\}$ of size p where $r_i = \{a_1^i, a_2^i, \ldots, a_{r-1}^i\}$ of the roots of $T_1, T_2, \ldots, T_m$.

5. Add p r-hyperedges $\{rt\} \cup r_i$ for $i = 1, \ldots, p$.

For $r \geq 3$, let r-pm_n be the number of r-perfect matchings on $[n]$. Let

$$E_{pm}^r(x) = \sum_{n \geq 0} r\text{-}pm_n \frac{x^n}{n!}$$

be the exponential generating function of the sequence r-pm_n. Let t_n^{r+1} be the number of rooted $(r+1)$-spanning trees on $[n]$ and let

$$T^{r+1}(x) = \sum_{n \geq 0} t_n^{r+1} \frac{x^n}{n!}$$

be the exponential generating function of the sequence t_n^{r+1}. It is easy to see from the above procedure that the following.

Theorem 7 *For all $r \geq 2$,*

$$T^{r+1}(x) = xE_{pm}^r(T^{r+1}(x))$$

4 r-Parking functions

In this section, we prove Theorem 3. We prove the theorem for $r = 2$. The proof is identical for higher values of r.

4.1 Connection to 3-spanning trees

Our proof of Theorem 3 closely mimicks that of Chebekin and Pylyavskyy [CP-05]. We first give a 2-parking function of length k from each 3-spanning tree on $2k + 1$ vertices arising from a fixed perfect matching.

We use the BFS ordering of vertices of $[n]$. We root the given 3-spanning tree at the vertex n and write the other vertices in the order of their "distance" from the root, breaking ties by the natural order $1 < 2 < \cdots < n$. For example, given the spanning tree in Figure 5, we redraw it rooted at vertex 7 as in the digram on the right of Figure 5 and order the vertices as $7 < 3 < 4 < 5 < 6 < 1 < 2$.

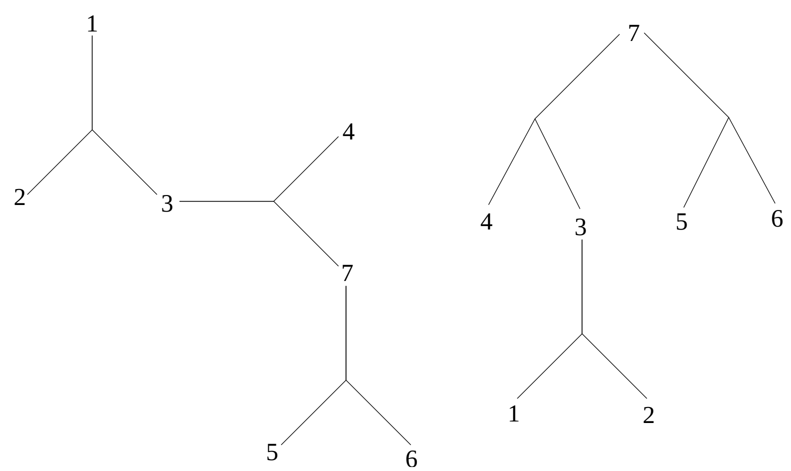

Figure 5: A spanning tree, redrawn with vertex 7 as the root.

Proof: (Of Theorem 3) We first deal with the case $r = 2$. For $n = 2k+1$, let $\mathcal{T}_k$ be the set of 3-spanning trees arising from the perfect matching $M = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}\}$. As there is a natural total order on the edges of this perfect matching, we label the edge $\{1, 2\}$ as the first edge, $\{3, 4\}$ as the second edge and so on. Given a 3-spanning tree T, let π_T be the BFS order on the vertices of T. The map π_T induces an order on the set of hyperedges containing a perfectly matched edge. For example, for the 3-spanning tree

of Figure 5, the order on the hyperedges containing $\{3,4\}$ is $\{3,4,7\} <_{\pi_T} \{3,4,5\} <_{\pi_T} \{3,4,6\} <_{\pi_T} \{1,3,4\} <_{\pi_T} \{2,3,4\}$. For the i-th edge of M, we refer to this order as $\pi_T(i)$.

We give a map $f_k : \mathcal{T}_k \mapsto \mathsf{Park}_k^2$ as follows. Let T be a 3-spanning tree arising from M. We give a number a_i for the i-th edge of M for each $1 \leq i \leq k$. The sequence $\overline{a} = (a_1, a_2, \ldots, a_k)$ is our candidate $f_k(T)$. Let a_i be the number of hyperedges which precede the unique hyperedge containing the i-th matched edge (ie $\{2i-1, 2i\}$) in the order $\pi_T(i)$.

In the 3-spanning tree of Figure 5, the sequence of numbers will be $(1,0,0)$ as $\{1,2,3\}$ is the 2-nd hyperedge in the order $\pi_T(1)$, $\{3,4,7\}$ is the first edge in the order $\pi_T(2)$ and $\{5,6,7\}$ is the first edge in the order $\pi_T(3)$.

The map f_k is clearly one-to-one. We first show that the sequences obtained satisfy the property that $f_k(i) \leq 2(i-1)$.

Let $\overline{b}$ be the weakly increasing rearrangement of the sequence $\overline{a}$. We recall that we obtained the sequence $\overline{a}$ from $T \in \mathcal{T}_k$. We denote by $\mathsf{ht}(i)$, the height of the i-th perfectly matched edge in the n-rooted version of T (in our example, $\mathsf{ht}(1) = 2, \mathsf{ht}(2) = 1, \mathsf{ht}(3) = 1$). Consider perfectly matched edges in increasing order of heights. It is clear that there is at least one perfectly matched edge (say edge i_1)whose height is 1. It is easy to check that $a_{i_1} = 0$ and hence $b_1 = 0 < 1$. The BFS order π_T induces an order on the matched edges of the tree according to the occurrence of the matched edge (or height). For example, in the tree of Figure 5, the order $\pi_T(M)$ on the matched edges is $\{3,4\} < \{5,6\} < \{1,2\}$. It is easy to see that the orders $\pi_T(M)$ and $\overline{b}$ are identical. Hence, the value of the i-th element of $\overline{b}$ is the hyperedge number of i-th matched edge in $\pi_T(M)$. Clearly, this value is maximised when the tree has only one hyperedge at each height when this value is at most $2(i-1)$.

We now show how to invert the map f_k. Let $\overline{a} \in \mathsf{Park}_k^2$. We identify the i-th coordinate with the i-th edge of the perfect matching. For $1 \leq j \leq k$, let $m(j)$ be the j-th perfectly matched edge. We recall the total order on the edges of the perfect matching $m(1) < m(2) < \cdots < m(k)$. Let $\overline{b} = (b_1 \leq b_2 \leq \cdots \leq b_k$ be the weakly increasing rearrangement of $\overline{b}$ satisfying $b_i = b_j$ implies $m(i) < m(j)$. We add hyperedges in increasing order of $\overline{b}$ to construct a 3-uniform spanning tree rooted at the vertex n. Since $b_1 = 0$, there is atleast one perfectly matched edge i_1 such that $a_{i_1} = b_1$ and the first hyperedge we add has the i_i-th perfectly matched edge and n (ie we add $\{2i_1 - 1, 2i_1, n\}$). If there are any more indices j such that $b_j = 0$, we add them too similarly. Thus we can assume that we are at iteration r with $b_r > 0$. Since $b_r \leq 2(r-1)$, and since we have added $r-1$ perfectly matched pairs before the r-th iteration, and since the vertex n also exists, we have $2(r-1)+1$ vertices already. We can consider the BFS order on the subtree consisting of just these $r-1$ hyperedges. Let the b_r-th vertex in the BFS be v_{b_r}. Let i_r be such that $a_{i_r} = b_r$. We add the hyperedge $\{2i_r - 1, 2i_r, v_{b_r}\}$. We do the same procedure till we get to index p such that $b_p > b_r$ (ie all of them are "connected" to the same vertex in the tree). Proceeding this way we get a tree $T \in \mathcal{T}_k$ as for each perfectly matched edge, we get a connection point which is already connected to the earlier tree. Let $g_k(\overline{a})$ be the tree obtained by this procedure. It is simple to check that $f_k(g_k(\overline{a})) = \overline{a}$. ∎

The proof for $r \geq 3$ is identical and following the proof above, we see that from $(r+1)$-spanning trees arising from a fixed r-perfect matching, we would get *r-parking functions*.

5 r-extended Shi hyperplane arrangements

In this subsection, we point out that the results obtained above and a theorem of Stanley [St-98], show that the number of $(r+1)$-spanning trees on $rk+1$ vertices arising from a fixed r-perfect matching is equal to the number of regions of the r-extended Shi arrangement S_k^r. We recall the following theorem from Stanley [St-98].

Theorem 8 (Stanley[St-98]) *The number of r-parking functions of length k is equal to the number of regions of the r-extended Shi arrangement S_k^r*

The proof of Theorem 4 is straightforward from the above theorem and Theorem 3. From [St-98], we see that the number of r-parking functions of length k is identical to the number of rooted r-forests on the vertex set k. From Theorem 3, we see that both are identical to the number of $(r+1)$-spanning trees on $rk+1$ vertices arising from a fixed r-perfect matching. We note that there is a simple bijection between such rooted r-forests on $[k]$ and $(r+1)$-spanning trees on $[rk+1]$ arising from a fixed r-perfect matching.

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